On the reduction of the interfacial stresses in a repaired beam with an adhesively FRP plate

**Kernou Nassim**¹, **Khalil Belakhdar**²

¹Département de génie civil et hydraulique, Université de Saida (Algérie) : nassimkernougeniecivil@yahoo.fr
²Département de génie civil et hydraulique, Université de Saida (Algérie) : be.khalil@gmail.com

**ABSTRACT.** Severe aging and deterioration of the transportation infrastructure has motivated extensive research in developing structural repair techniques using FRP or steel plates. As a result, previous researchers have developed several analytical methods to predict the interface performance of bonded repairs. An important feature of the reinforced beam is the significant stress concentration in the adhesive at the ends of FRP plate. In this paper, a numerical solution using finite difference method is used to calculate the interfacial stress distribution in beams strengthened with FRP plate having a tapered ends. These latter, can significantly reduce the stress concentrations. Numerical results from the present analysis are presented to demonstrate the advantages of using the tapers in design of strengthened beams.

**KEY WORDS:** Plate bonding; FRP composite; Interfacial stresses; Repaired beam; Design; Taper

1. Introduction

Over the past several decades, extensive research and development in the field of materials engineering and science have been carried out with fibre-reinforced plastic (FRP) composites leading to a wide range of practical applications [1, 2]. As a nation’s infrastructure ages, one of the major challenges the construction industry faces is that the number of deficient structures continues to grow. The applications of using externally bonded FRP plates to reinforced concrete (RC) or steel structures have shown that the technique is sound and efficient and offers a practical solution to this pressing problem. Retrofitting using externally bonded plates is quick, easy with respect to material handling, causes minimal site disruption and produces only little changes in section size. In recent years, many studies have been carried out on the behaviour and strength of such retrofitting method [3 – 15]. However, the disadvantage of this technique is that the strengthened members are susceptible to stresses concentration near the plate end, which may cause a total debonding between the member and the FRP plate. Some researchers note that reducing the FRP or steel plate thickness near the plate end is an effective method to minimize the interfacial stresses [10].

As a further development of the solutions by Stratford [10], Smith [9] and Tounsi [12], this paper presents a simple numerical solution for obtaining the shear and the normal stresses in the adhesive layer of a retrofitted beam under externally loads. The method can be used to design strengthened beams with section properties that change along the beam such as tapered plates. Finally, a parametric study was conducted to compare the results of models with different geometries.

2. Theoretical approach

The derivation of the present solution is described in terms of adherends 1 and 2, where adherend 1 is the beam and adherend 2 is the soffit plate (Fig 1). The assumptions adopted in the present solution are summarized below:

1. The beam, adhesive, and FRP materials behave elastically, linearly, and isotropically.
2. The shear stress in the interface is proportional to the shear slip.
3. Since the thickness of the adhesive layer is small, both the shear and peeling stresses in the adhesive are assumed constant across its thickness.

4. The two bonded bodies have the same bending curvature at the same section. This assumption is not made in the peel stress solution.

**Fig. 1.** Simply supported beam strengthened with bonded FRP plate

### 2.1. Equilibrium and constitutive relationships in the beam and plate

A differential segment of plated beam is shown in Fig. 2, where the interfacial shear and normal stresses are denoted by \( \tau(x) \) and \( \sigma(x) \), respectively.
Fig. 2 also shows the positive sign convention for the bending moment, shear force, axial force and applied loading. Consideration of equilibrium gives:

\[ N_b = -N_p \quad \text{and} \quad M_b = N_p z - M_p \]  \hspace{1cm} (1)

Where \( z \) is the lever arm between the centroids of the plate and the beam

\[ z = \frac{t_b}{2} + t_a + \frac{t_p}{2} \]  \hspace{1cm} (2)

Where \( t_b \), \( t_p \) and \( t_a \) are the depth of beam, plate and the thickness of adhesive. The subscripts \( b, p \) and \( a \) denote beam, plate and adhesive, respectively.

The strains at the base of the beam and the top of the plate are given as:

\[ \varepsilon_b(x) = -\frac{t_b}{2E_b I_b} M_b(x) + \frac{N_b(x)}{E_b A_b} \]  \hspace{1cm} (3)

\[ \varepsilon_p(x) = \frac{t_p}{2E_p I_p} M_p(x) + \frac{N_p(x)}{E_p A_p} \]  \hspace{1cm} (4)
Where \( E \) is the elastic modulus, \( A \) the cross-sectional area, and \( I \) is the second moment of area.

The curvature \( \psi \) in the beam and the plate can be expressed as:

\[
\psi_p = \frac{M_p}{E_p I_p}, \quad \psi_b = \frac{M_b}{E_b I_b} = \frac{N_p z - M_p}{E_b I_b}
\]

**2.2. Equilibrium and constitutive relationships across the adhesive joint**

The mean shear stress \( \tau(x) \) and peel stress \( \sigma(x) \) acting across the adhesive joint are assumed to act about the mid–plane of the adhesive layer. By considering equilibrium of a short length of the plate (Fig. 2):

\[
\tau(x) = \frac{1}{b_a} \frac{dN_p}{dx}
\]

Where \( b_a \) is the width of the adhesive layer.

\[
V_p = -\frac{dM_p}{dx} + \frac{t_p}{2} \frac{dN_p}{dx}
\]

\[
\sigma(x) = \frac{1}{b_a} \frac{dV_p}{dx}
\]

The constitutive response of the adhesive is described by

\[
\gamma_a = \frac{\tau(x)}{G_a} = \frac{1}{b_a G_a} \frac{dN_p}{dx}, \quad \epsilon_a = \frac{\sigma(x)}{E_a}
\]

**2.3. Shear compatibility of adhesive layer**

The shear stress, \( \tau(x) \), across the adhesive interface is determined by examining shear compatibility across the adhesive joint. The shear strain \( \gamma_a \) in the adhesive layer can be written as

\[
\gamma_a = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\]

Where \( u \) and \( v \) are the horizontal and vertical displacements respectively at any point in the adhesive layer.

Assuming that the displacement field varies linearly through the adhesive layer, the average shear strain in the adhesive layer is given by:

\[
\gamma_a = \frac{u_p - u_b}{t_a} + \frac{1}{2} \left( \frac{d v_p}{dx} + \frac{d v_b}{dx} \right)
\]

Where \( u_p \) and \( u_b \) are the longitudinal displacements at the top of the plate and the base of the beam, respectively.

Differentiating the above expression with respect to \( x \) gives:

\[
t_a \frac{d\gamma_a}{dx} = \epsilon_p - \epsilon_b + \frac{t_a}{2} (\psi_p + \psi_b)
\]
Where $\psi = d^2v/dx^2$ is the curvature in the beam and the plate.

Substituting for the strains using the constitutive relationships (Eqs. (1), (3), (4), (5) and (9)) gives the compatibility equation for shear strain across the adhesive interface

$$\frac{t_a}{b_a G_a} \frac{d^2 N_p}{dx^2} = \left( \frac{1}{E_p A_p} + \frac{1}{E_b A_b} \right) N_p + \frac{t_p + t_a}{2E_p I_p} M_p + \frac{t_b + t_a}{2E_b I_b} (N_p z - M_p)$$ (13)

The unknown variables, $N_p$ and $M_p$ are coupled. To simplify the solution, the difference between the beam and plate curvatures is assumed negligible for the purposes of determining $N_p$:

$$\psi_p = \frac{M_p}{E_p I_p} = \frac{N_p z - M_p}{E_b I_b} = \psi_b$$ (14)

Hence

$$\psi_p = \psi_b = \frac{N_p z}{E_p I_p + E_b I_b}$$ (15)

$M_p$ can now be eliminated from Eq. (13), giving the governing equation for $N_p$

$$-f_1 \frac{d^2 N_p}{dx^2} + f_2 N_p = 0$$ (16)

Where

$$f_1 = \frac{t_a}{G_a b_a}, \quad f_2 = \frac{1}{E_p A_p} + \frac{1}{E_b A_b} + \frac{z^2}{E_p I_p + E_b I_b}$$ (17)

2.4. Normal adhesive stress

The normal stress $\sigma(x)$ is found by examining the through-thickness strain, $\epsilon_a$ in the adhesive, the difference in vertical displacement across the adhesive joint:

$$\epsilon_a = \frac{\partial v}{\partial y} = \frac{v_p - v_b}{t_a}$$ (18)

Rewriting in terms of stress and differentiating twice with respect to $x$

$$\frac{t_a}{E_a} \frac{d^2 \sigma}{dx^2} = \psi_p - \psi_b$$ (19)

Substituting for the curvatures using Eq. (5) gives the compatibility equation for normal stress:

$$\frac{t_a}{E_a} \frac{d^2 \sigma}{dx^2} = \frac{M_p}{E_p I_p} - \frac{N_p z - M_p}{E_b I_b}$$ (20)

Using Eqs. (7) and (8), the equation (20) can be rewritten as
31èmes Rencontres de l’AUGC, E.N.S. Cauchan, 29 au 31 mai 2013

\[
a_1 \frac{d^4 M_p^*}{dx^4} + a_2 M_p^* = a_3 N_p^*
\]

(21)

Where

\[
a_1 = \frac{t_a}{E_a b_a}, \quad a_2 = \frac{1}{E_p I_p} + \frac{1}{E_b I_b}, \quad a_3 = \frac{z - t_p / 2}{2E_p I_p} - \frac{t_p}{2E_p I_p}
\]

(22)

And \( M_p^* \) is the transformed moment about the plate – adhesive interface,

\[
M_p^* = M_p - \frac{t_p}{2} N_p
\]

(23)

3. Interfacial stresses with taper

The governing differential equations for shear stress (Eq. (16)) and normal stress (Eq. (21)) are valid for plated beams with geometric and material properties that vary along its length, however, the coefficients \( f_2, a_2 \) and \( a_3 \) vary in \( x \), and a closed form solution is not possible. A finite – difference method can be used to find the adhesive stresses for cases with varying section properties.

A finite – difference solution using constant node spacing, \( \Delta \), is outlined below. The nodes are numbered \( i = 1 \ldots n \) (from \( x = 0 \) to \( x = L/2 \), the centre of the beam). Virtual nodes (-1, 0, \( n + 1, n + 2 \)) are used to allow derivatives to be defined at nodes 1 and \( n \). Superscripts are used to define node numbers in the following equations.

3.1. Shear stress with taper

The governing equation for the plate force (Eq. (16)) can be written at each node \( (i = 1 \ldots n) \) along the beam:

\[
-f_1 N_p^{i-1} - 2N_p^i + N_p^{i+1} + f_2^i N_p^i = 0
\]

(24)

Considering the boundary conditions:

1. Due to symmetry, the shear stress at mid – span is zero \( (\tau = dN_p / dx = 0) \), i.e.

\[
\frac{N_p^{n+1} - N_p^{n-1}}{2\Delta} = 0
\]

(25)

2. The axial force in the plate at position \( x = 0 \) is zero.

\[
N_p^1 = N_p^1 \big|_{x=0} = 0
\]

(26)

These \( n + 2 \) simultaneous equations are solved explicitly to find the plate force at each of the \( n + 2 \) nodes. An implicit solution method (for example, by varying the shear stress at node 1 so as to satisfy boundary condition (25) at node \( n \)) will not be successful, due to the sensitivity of conditions at node \( n \) to small changes at node 1.
The shear stress distribution follows from Eq. (6):

\[ \tau_i = \frac{1}{b_p} \frac{N_i^{i+1} - N_i^{i-1}}{2\Delta} \] (27)

3.2. Normal stress with taper

A fourth–order finite–difference solution is required to find the transformed plate moment. The governing equation for normal stress (Eq. (21)) is written at each node \((i = 1 \ldots n)\) along the beam:

\[ a_1 M_p^{* (i-2)} - 4M_p^{* (i-1)} + 6M_p^{* (i)} - 4M_p^{* (i+1)} + M_p^{* (i+2)} \Delta^2 + a_i^1 M_p^{* i} = a_i^1 N_p^{i} \] (28)

At the end of plate, the bending moment and shear force are known. Hence, the boundary conditions can be written as:

1. Boundary condition (1): the applied bending moment and the axial force at \(x = 0\) are zero.

\[ M_p^{*1} = M_p^{*1} \bigg|_{x=0} = 0 \] (29)

2. Boundary condition (2): \( V_p = dM_p^{*} / dx = V_p \bigg|_{x=0} \) at \(x = 0\)

\[ \frac{M_p^{* (c-2)} - 8M_p^{* (c-1)} + 8M_p^{* (c+1)} + M_p^{* (c+2)}}{12\Delta} = V_p \bigg|_{x=0} \] (30)

3. Boundary condition (3): \( V_p = dM_p^{*} / dx = 0 \) at the centre of a symmetric beam:

\[ \frac{M_p^{* (n-2)} - 8M_p^{* (n-1)} + 8M_p^{* (n+1)} + M_p^{* (n+2)}}{12\Delta} = 0 \] (31)

4. Boundary condition (4): \( d^3 M_p^{*} / dx^3 = 0 \) at the centre of a symmetric beam:

\[ \frac{-M_p^{* (n-2)} + 2M_p^{* (n-1)} - 2M_p^{* (n+1)} + M_p^{* (n+2)}}{2\Delta} = 0 \] (32)

These \(n + 4\) simultaneous equations are solved explicitly for the plate force at each of the \(n + 4\) nodes. The normal stress distribution follows Eqs. (7), (8) and (23):

\[ \sigma_i = \frac{1}{b_p} \left[ \frac{-M_p^{* (i-2)} + 16M_p^{* (i-1)} - 30M_p^{* i} + 16M_p^{* (i+1)} - M_p^{* (i+2)}}{12\Delta^2} \right] \] (33)
4. Results and discussion

4.1. FRP Plate with constant thickness (without taper)

Finite element (FE) analysis has been employed to validate the results of the above procedure. A quadratic brick element is employed in the present calculation. The half of beam is analyzed because of the symmetry. The refined mesh is arranged near the end of the FRP plate.

The geometric and material properties of the beam are shown in Fig. 3 and table 1. A constant CFRP-plate thickness is analyzed firstly using finite difference method and FE method. The beam was subjected to two cases of loading

1. Uniformly distributed load (UDL) q=500kN/m²
2. Mid–point load P=500kN

The obtained results are compared with an analytical solution carried out by Smith and Teng [9]. Figs. 4 and 5 show the interfacial stresses vs. the distance from the FRP plate end. It is shown that both finite difference solution (FD) and FE calculation are agreeable well with the analytical curve. There are stress concentrations of normal and shear stresses at the end of FRP plate, which cause the interfacial delamination or failure.

![Fig. 3: Geometric properties of the composite beam](image)

### Table 1 Dimensions and material properties of the composite beam

<table>
<thead>
<tr>
<th>Component</th>
<th>Width [mm]</th>
<th>Depth [mm]</th>
<th>Young’s modulus [GPa]</th>
<th>Poisson’s ratio</th>
<th>Shear modulus [Gpa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam</td>
<td>211.9</td>
<td>544.9</td>
<td>210</td>
<td>0.3</td>
<td>-</td>
</tr>
<tr>
<td>Adhesive layer</td>
<td>211.9</td>
<td>2</td>
<td>10</td>
<td>0.3</td>
<td>3.7</td>
</tr>
<tr>
<td>FRP plate</td>
<td>211.9</td>
<td>12</td>
<td>310</td>
<td>0.3</td>
<td>3.7</td>
</tr>
</tbody>
</table>
Fig. 4: Comparison of interfacial shear and normal stress for a beam with a bonded FRP soffit plate subjected to a UDL.

Fig. 5: Comparison of interfacial shear and normal stress for a beam with a bonded FRP soffit plate subjected to a mid–point load.

4.2. FRP Plate with taper

Herein the effect of FRP plate with a generally variable thickness (Fig. 6) is presented. The thickness of the FRP plate $t_p = t_p(x)$ is described by arbitrary function of the longitudinal coordinate $x$; hence the cross-sectional area and the second moment of area of the FRP plate, $A_p(x)$ and $I_p(x)$, are also functions of the coordinate $x$. Four particular cases are examined.

(a) Type A: constant thickness
\[ t_p(x) = t_p \text{ for } 0 \leq x \leq \frac{L}{2} \]  

\[(b) \text{ Type B: linear variation} \]

\[ t_p(x) = \left(\frac{t_p-t_e}{a}\right)x + t_e \text{ for } 0 \leq x \leq a \]  
\[ t_p(x) = t_p \text{ for } a \leq x \leq \frac{L}{2} \]  

\[(c) \text{ Type C: Tangent – parabolic variation} \]

\[ t_p(x) = \left(\frac{t_p-t_e}{a^2}\right)x^2 + t_e \text{ for } 0 \leq x \leq a \]  
\[ t_p(x) = t_p \text{ for } a \leq x \leq \frac{L}{2} \]  

\[(d) \text{ Type D: Parabolic variation} \]

\[ t_p(x) = -\left(\frac{t_p-t_e}{a^2}\right)x^2 + 2\left(\frac{t_p-t_e}{a}\right)x + t_e \text{ for } 0 \leq x \leq a \]  
\[ t_p(x) = t_p \text{ for } a \leq x \leq \frac{L}{2} \]

Fig. 6: Thickness patterns of FRP plate.

The FRP plate with constant thickness (type A) will be regarded as the reference case. A numerical study is then conducted to supply information on the contribution of the shape of the thickness profile on the reduction of edge interfacial stresses. In this study, the length of the taper is \( a = 200 \text{ mm} \) and the thickness at the end of the tape is \( t_e = 2 \text{ mm} \). From the results presented in table 2, we can observe that the decrease of the thickness of the
FRP plate in the edge region leads to a reduction in the edge stresses. We can also conclude that FRP plate of type C gives an important reduction in edges interfacial stresses. Figures 7 and 8 describe the typical stress field to clarify the comparison of the results with those of the reference case (FRP plate with constant thickness). Hence, reducing the thickness of the FRP plate in the edge region may be considered as an effective way for reducing the magnitude of the edge stresses involved.

Table 2 Interfacial edge stresses for different shape of taper

<table>
<thead>
<tr>
<th>Type</th>
<th>Max shear stress $\tau$ [MPa]</th>
<th>Max normal stress $\sigma$ [MPa]</th>
<th>$\left(\tau_{\text{Type A}} - \tau\right)%$</th>
<th>$\left(\sigma_{\text{Type A}} - \sigma\right)%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type A</td>
<td>13.999</td>
<td>10.430</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Type B</td>
<td>8.718</td>
<td>4.643</td>
<td>37.72%</td>
<td>55.48%</td>
</tr>
<tr>
<td>Type C</td>
<td>7.181</td>
<td>3.666</td>
<td>48.70%</td>
<td>64.85%</td>
</tr>
<tr>
<td>Type D</td>
<td>9.988</td>
<td>5.530</td>
<td>28.65%</td>
<td>46.98%</td>
</tr>
</tbody>
</table>

Fig. 7: Interfacial shear stress for various thickness profiles of FRP plate.
4.3. Parametric study

Various parameters influence the maximum values of the shear and normal stresses in the bonding region. In this study we used the taper of type C. For retrofitted beams, the most important ones are the thickness and shear modulus of the adhesive, and the thickness and the elastic modulus of the FRP plate. These parameters were studied by several authors [7, 11 to 14]. In the present study, we intend to show how the maximum adhesive stresses are influenced by the dimension of the taper. The important parameters of the taper are: the length of the taper \(a\) and the thickness at the end of the taper \(t_e\). Figures 9 and 10 plot the interfacial stresses at the tapered end of the plate versus the length of the taper and the thickness end, respectively. The parametric study indicates the beneficial effect of having a thin tapered end and a long taper. For the latter, the benefit appears to have saturated when the length of the taper is beyond 500 mm.
Fig. 9: Interfacial edge stresses for different length of the taper ($a = 2$ mm).

Fig. 10: Interfacial edge stresses for different tapered end thicknesses ($a = 200$ mm).

5. Conclusion
The present study has developed a relatively simple procedure for controlling the high shear and normal stress concentrations that occur at the edges of the FRP plate in externally strengthened beams. FE analysis has been employed to validate the results from the analytical and the numerical solutions, and the agreement between the results obtained from the different solutions is good, which demonstrates that present procedure is simple yet accurate.

High stress concentrations occur at the free ends of adhesively bonded plates. The taper, however, reduce the maximum shear stresses by about 40% and the maximum normal stresses by about 60%. In addition, it has been shown that the shape of the taper has an important effect on the reduction of such stresses. Hence, the taper is very beneficial for avoiding debonding of the FRP plates from the beams. The dimensions of the taper also have an influence on the interfacial edge stresses. The maximum shear and normal stresses decrease as the thickness of the end of the taper decreases and the length of the taper increases. However, there is no further change in stress if the length of the taper is increased beyond 500 mm.

References

